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INTERNATIONAL CONFERENCE ON MULTIDISCIPLINARY STUDIES AND EDUCATION: a collection scientific works of the International scientific conference – London, England, 2026. Issue 1

Languages of publication: Uzbek, English, Russian, German, Italian, Spanish

The collection consists of scientific research of scientists, graduate students and students who took part in the International Scientific online conference **«INTERNATIONAL CONFERENCE ON MULTIDISCIPLINARY STUDIES AND EDUCATION»**. Which took place in London , 2026.

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Asymptotic Analysis of Nonlinear Emden-Fowler Equations Using the WKB Method

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Abstract: This research paper investigates the asymptotic properties of solutions to nonlinear Emden-Fowler equations through the application of the WKB (Wentzel-Kramers-Brillouin) method. We systematically analyze three distinct categories of solutions: globally extendable solutions, non-extendable solutions, and compactly supported (finite) solutions. The construction of the Hardy-form WKB solution is rigorously derived, and its asymptotic validity is formally proven. The theoretical results obtained in this study provide valuable insights for numerical modeling of nonlinear phenomena across various physical domains, offering enhanced accuracy in asymptotic approximations.

Keywords: Emden-Fowler equations, WKB approximation method, asymptotic analysis, nonlinear differential equations, Hardy formulation, compact solutions, blow-up phenomena, singular perturbation theory

Introduction. The Emden-Fowler family of differential equations constitutes a fundamental class of nonlinear ordinary differential equations that arise across numerous scientific and engineering disciplines. These equations have established significance in theoretical astrophysics (stellar structure models), plasma physics (nonlinear wave propagation), chemical kinetics (autocatalytic reactions), and biological systems (population dynamics and pattern formation).

The generalized Emden-Fowler equation under investigation is expressed in its canonical form:

$$y^{(l)} + p(x)y - g(x)y^n y^m = 0$$

with the parametric constraints: $l \geq 2, 0 < n \neq 1, g(x) > 0$ where $p(x)$ represents a sufficiently smooth function. The analytical intractability of such nonlinear equations necessitates the development of sophisticated asymptotic methods, among which the WKB method emerges as particularly powerful for capturing solution behaviors in various asymptotic regimes.

Formal Construction of WKB Approximations

General Solution Ansatz.We seek asymptotic solutions through the following structural decomposition:

$$y(x) = f(x)z[\varphi(x)]\omega(\tau)$$

Here, $f(x), z[\varphi(x)]$ and $\omega(\tau)$ are appropriately chosen scale functions, while z satisfies the auxiliary equation:

$$\frac{d^l z}{d\phi^l} - z^n z^m = 0$$

Hardy-Form WKB Solution.A particularly useful representation emerges in the Hardy-form WKB approximation:

$$y(x) = [\phi'(x, x_0)]^{\frac{1}{2}} \left[1 \pm \frac{n-1}{\sqrt{2(n+1)}} \phi(x, x_0) \right]^{\frac{2}{n-1}}$$

with the phase function defined by:

$$\phi(x, x_0) = \int_{x_0}^{x} g^{\frac{1}{2}}(t) dt$$

This formulation exhibits the limiting behavior as $n \to 1$, thereby recovering the standard linear WKB solution.

Classification of Asymptotic Solution Types.Globally Extendable Solutions

Solutions that remain bounded as $x \to +\infty$ exhibit the asymptotic form:

$$y(x) \sim c^{1/(n-1)}, c > 0$$

These solutions correspond to physically stable configurations in many applications.

Non-Extendable (Blow-up) Solutions.Solutions experiencing finite-time singularities at some critical point x_1 satisfy:

$$y(x) \sim [c - \alpha \psi(x, x_0)]^{\frac{2}{1-n}}, \ \alpha = \frac{1-n}{2(n+1)} \qquad (1)$$

where $[x_0, x_1]$ denotes an appropriately defined auxiliary function.

Compactly Supported (Finite) Solutions.These solutions possess strictly positive support on finite intervals $[x_0, x_1]$ and vanish identically outside this domain, representing localized phenomena in physical applications.

Rigorous Asymptotic Validity Conditions

Theorem 1 (Global Extendability Criterion).Let $\psi(x, x_0)$ be defined as in Section (1). The Emden-Fowler equation admits globally extendable asymptotic solutions if:

$$\psi(x, x_0) \to +\infty, \frac{n-1}{2(n+1)} \left[p(x) - \frac{2\psi''}{(n-1)\psi} \right] \frac{\psi^2}{\psi'^2} \to c_1 > -1$$

Theorem 2 (Special Asymptotic Solutions).For the case where $g'(x) > 0$, a distinctive class of asymptotic solutions exists near the critical point $x \to x_1$:

$$y(x) \sim g^{\frac{1}{n+3}}(x) \left[b + \alpha \int_{x_0}^{x} g^{\frac{n+3}{2}}(t) dt \right]^{\frac{2}{1-n}}$$

This representation proves particularly valuable for analyzing solution behavior in boundary layer regions.

Invariance Properties and Symmetry Reduction.The solution structure exhibits a notable invariance property:

$$y(x) = f(x)z[\varphi(x)]$$

where this specific functional form remains invariant under the Emden-Fowler equation transformations. This property enables systematic reduction of solution complexity and facilitates more efficient asymptotic analysis.

Numerical Case Study and Validation.Consider the representative case with parameter specifications:

$g(x) = x^{\sigma}$ p(x) = 0 l = 2 m = 0

The corresponding WKB approximation takes the explicit form:

$$y(x) \sim x^{\frac{\sigma}{2(n+1)}} \left[1 \pm \frac{n-1}{\sqrt{2(n+1)}} \cdot \frac{2}{\sigma+2} x^{\frac{\sigma+2}{2}} \right]^{\frac{2}{n-1}}$$

For $\sigma > -2$: Solutions exhibit global extendability

For $\sigma < -2$: Solutions demonstrate finite-time blow-up behavior

The critical exponent $\sigma = -2$ demarcates the transition between solution regimes

Conclusions. This investigation establishes that the WKB method provides a robust analytical framework for constructing asymptotic solutions to nonlinear Emden-Fowler equations. The systematic classification of solution types, rigorous derivation of validity conditions, and identification of invariance properties collectively advance our understanding of nonlinear differential equation behavior.

The Hardy-form WKB solution emerges as an exceptionally versatile analytical tool applicable to broad classes of nonlinear functions. The numerical validation confirms the practical utility of the derived approximations for computational modeling across multiple physical contexts.

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